Moment of Inertia -

\[ A = \int dA \]

\[ B_x = \int y dA, \quad B_y = \int x dA \]

First moment of area about \( x \) \& \( y \) axis

\[ \bar{x} = \frac{\int x dA}{\int dA} = \frac{B_y}{A} \]

\[ \bar{y} = \frac{\int y dA}{\int dA} = \frac{B_x}{A} \]

\[ \bar{x} = \frac{A_1 x_1 + A_2 x_2 + A_3 x_3}{A_1 + A_2 + A_3} \]

\[ \bar{y} = \frac{A_1 y_1 + A_2 y_2 + A_3 y_3}{A_1 + A_2 + A_3} \]

\( x_1, x_2, x_3 \) \& \( y_1, y_2, y_3 \) are distance from reference axis \( AA \) \& \( BB \) respectively.

\( \bar{x}, \bar{y} \) also come w.r.t. axis \( AA \).

We can assume this reference axis \( AA \) \& \( BB \) anywhere.

For ease of computation, we generally fix the axes \( AA \) \& \( BB \) as shown in fig.
Moment of Inertia of Areas / Second Moments of area

\[
I_x = \int y^2 dA
\]

\[
I_y = \int x^2 dA
\]

\[
I_{xx} = \int y^2 b dy = \frac{bh^3}{12}
\]

\[
I_{yy} = \int x^2 h dx = \frac{hb^3}{12}
\]

\[
I_{bb} = \int y^2 dA = \int y^2 b dy = \frac{bh^3}{3}
\]

Moment of inertia increases as the reference axis is moved parallel to itself further from the centroid.

\[
I_x = \frac{bh^3}{12} - \frac{b_1 h_1^3}{12}
\]

\[
I_y = ?
\]
**Parallel Axis Theorem**

\[ I_{Ax} = I_{CoM} + AR^2 \]

- Moment of Inertia about a axis AA, that is parallel to CoM axes and at a distance of R

see in fig. (1)

\[ I_{AB} = I_{XX} + A \cdot (h/2)^2 \]

\[ = \frac{bh^3}{12} + \frac{(bh)(h/2)^2}{4} \]

\[ = \frac{bh^3}{12} + \frac{bh^3}{4} \]

\[ = \frac{bh^3}{3} \]

**Fig. (2)**

\[ I_y = \frac{hb^3}{12} - \frac{h_1 b_1^3}{12} \]

**Fig. (3)**

\[ I_y = \frac{hb^3}{12} - \left( \frac{h_1 b_1^3}{12} + h_1 b_1 \left( \frac{b_2 - b_1}{2} \right)^2 \right) \]

**Fig. (4)**

\[ I_y = \frac{hb^3}{12} - \frac{2}{5} \left( \frac{h_1 (b_2)^3}{12} + h_1 b_1 \left( b_2 - \frac{b_1}{2} \right)^2 \right) \]

**Polar Moment of Inertia**

\[ I_p = I_{pC} + Ad^2 \]

- Distance d = distance b/w OA and C

\[ I_p = \int (x^2 + y^2) \, dA \]

**Polar Moment of Inertia for Positive Axes Perpendicular to the Plane of Area**

\[ I_p = I_x + I_y \]

\[ I_z = I_x + I_y \]
The product of inertia of an area is zero with any pair of axes in which one axis (or both) is an axis of symmetry.

**Products of Inertia**

\[ I_{xy} = \int xy \, dA \]

**Parallel Axis Theorem for Product of Inertia**

\[ I_{xy} = I_{xy}^c + Ad^2 \]

\( d \) and \( d^2 \) are coordinates of centroid w.r.t. \( xy \) axes, \( (d \) and \( d^2 \) may be +ve or -ve).

\[ dI_{xy} = [dA] \cdot x \cdot dy \]

\[ dI_{xy} = \int dy \cdot \left( \frac{b}{h} \cdot x(h-y) \right) \cdot \frac{b(h-y)}{2h} \cdot y \]

\[ I_{xy} = \int_0^h dI_{xy} \]

\[ I_{xy} = \frac{b^2h^2}{24} \]

**Rotation of Axes**

\[ I_{x'y'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y \cos 2\theta + I_y \sin 2\theta}{2} \]

\[ I_{x'y'} = \frac{I_x - I_y \sin 2\theta + I_y \cos 2\theta}{2} \]

\[ I_{x'y'} = \frac{I_x + I_y}{2} \]

\[ \tan 2\theta = -\frac{I_y}{I_x - I_y} \]

Peer: "You might consider using different colors for text, changing font sizes, and adjusting spacing to make your notes more legible."
\[ I_1, I_2 = \frac{I_x + I_y}{2} \pm \sqrt{(\frac{I_x - I_y}{2})^2 + I_z^2} \]

- Principal moment of inertia.

- Polar moment of inertia is constant.

\[ I_1 + I_2 = I_x + I_y \]

Radius of gyration:

\[ r_x = \sqrt{\frac{I_x}{A}}, \quad r_y = \sqrt{\frac{I_y}{A}}. \]

Radius of gyration of an area is the distance from the axis at which the entire area could be Concentrated and still have the same moment of inertia as the original area.

- Slenderness ratio: \( \lambda = \frac{r_{eff}}{r_{min}} \).
1. A square hole is punched out of a circular laminate, as shown in fig. Find the moment of inertia about Y-Y axis through e.g.

2. Find moment of inertia of the section shown in fig. below about X-X axis.

3. Find the moment of inertia of the triangular section shown in Fig.

4. For the I section shown in figs, determine the position of centroid and moment of inertia about the base flange ($I_a$).
5. The combined angle and channel section shown in the fig. forms part of a crane runway beam. For that calculate:

(i) Coordinate of the centroid
(ii) Second moment of area about X - X
(iii) Second moment of area about X - X
(iv) Product of inertia about O

[SSC JE - 2011]
Properties of Plane areas

Notation: 
- $A =$ area
- $\bar{x}, \bar{y} =$ distances to centroid $C$
- $I_x, I_y =$ moments of inertia with respect to the $x$ and $y$ axes, respectively
- $I_{xy} =$ product of inertia with respect to the $x$ and $y$ axes
- $I_p = I_x + I_y =$ polar moment of inertia
- $I_{BB} =$ moment of inertia with respect to axis $B-B$

**Rectangle** (Origin of axes at centroid.)

$$A = bh \quad \bar{x} = \frac{b}{2} \quad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{12} \quad I_y = \frac{hb^3}{12}$$

$$I_{xy} = 0 \quad I_p = \frac{bh}{12} (h^2 + b^2)$$

**Triangle** (Origin of axes at centroid.)

$$A = \frac{bh}{2} \quad \bar{x} = \frac{b + c}{3} \quad \bar{y} = \frac{h}{2}$$

$$I_x = \frac{bh^3}{36} \quad I_y = \frac{bh}{36} (b^2 - bc + c^2)$$

$$I_{xy} = \frac{bh^2}{72} (b - 2c) \quad I_p = \frac{bh}{36} (h^2 - b^2 - bc + c^2)$$

**Isosceles triangle** (Origin of axes at centroid.)

$$A = \frac{bh}{2} \quad \bar{x} = \frac{b}{2} \quad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \quad I_y = \frac{bh}{48} \quad I_{xy} = 0$$

$$I_p = \frac{bh}{144} (4h^2 + 3b^2) \quad I_{BB} = \frac{bh^3}{12}$$

(Note: For an equilateral triangle, $h = \sqrt{3}b/2$.)

**Right triangle** (Origin of axes at centroid.)

$$A = \frac{bh}{2} \quad \bar{x} = \frac{b}{3} \quad \bar{y} = \frac{h}{3}$$

$$I_x = \frac{bh^3}{36} \quad I_y = \frac{hb^3}{36} \quad I_{xy} = -\frac{b^2h^2}{72}$$

$$I_p = \frac{bh}{36} (h^2 + b^2) \quad I_{BB} = \frac{bh^3}{12}$$
Trapezoid (Origin of axes at centroid.)
\[ A = \frac{h(a + b)}{2} \quad \bar{y} = \frac{h(2a + b)}{3(a + b)} \]
\[ I_x = \frac{k(a^2 + 4ab + b^2)}{36(a + b)} \quad I_{yy} = \frac{k(3a + b)}{12} \]

Circle (Origin of axes at center.)
\[ A = \pi r^2 \quad I_x = I_y = \frac{\pi r^4}{4} = \frac{nd^4}{64} \]
\[ I_{yy} = 0 \quad I_x = \frac{\pi r^4}{2} = \frac{nd^4}{32} \quad I_{yy} = \frac{5\pi r^4}{4} = \frac{5nd^4}{64} \]

Parabolic semisegment (Origin of axes at corner.)
\[ y = f(x) = h \left( 1 - \frac{x^2}{b^2} \right) \]
\[ A = \frac{2bh}{3} \quad \bar{x} = \frac{3b}{8} \quad \bar{y} = \frac{2h}{5} \]
\[ I_x = \frac{16bh^3}{105} \quad I_y = \frac{2bh^3}{15} \quad I_{yy} = \frac{b^2h^2}{12} \]

Parabolic spandrel (Origin of axes at vertex.)
\[ y = f(x) = \frac{hx^2}{b^2} \]
\[ A = \frac{bh}{3} \quad \bar{x} = \frac{3b}{4} \quad \bar{y} = \frac{3h}{10} \]
\[ I_x = \frac{bh^3}{21} \quad I_y = \frac{bh^3}{5} \quad I_{yy} = \frac{b^2h^2}{12} \]

Semisegment of nth degree (Origin of axes at corner.)
\[ y = f(x) = h \left( 1 - \frac{x^2}{b^2} \right)^n \quad n > 0 \]
\[ A = bh \left( \frac{n}{n + 1} \right) \quad \bar{x} = \frac{bh}{2(n + 2)} \quad \bar{y} = \frac{bh}{2n + 1} \]
\[ I_x = \frac{2bh^3}{(n + 1)(2n + 1)(3n + 1)} \quad I_y = \frac{bh^3}{(n + 1)(2n + 1)(3n + 1)} \quad I_{yy} = \frac{b^2h^2}{4n + 1} \]

Spandrel of nth degree (Origin of axes at vertex.)
\[ y = f(x) = \frac{hx^n}{b^n} \quad n > 0 \]
\[ \bar{x} = \frac{bh}{n + 1} \quad \bar{y} = \frac{bh}{n + 2} \quad \bar{y} = \frac{bh}{2(n + 1)} \]
\[ I_x = \frac{bh^3}{(3n + 1)} \quad I_y = \frac{bh^3}{n + 3} \quad I_{yy} = \frac{b^2h^2}{4(n + 1)} \]

Sine wave (Origin of axes at centoid.)
\[ A = 4bh \quad \bar{y} = \frac{\pi b}{8} \]
\[ I_x = \frac{8}{9\pi} (bh^3 \approx 0.08659bh^3) \quad I_y = \left( \frac{4}{\pi} - \frac{32}{9\pi} \right) bh^3 \approx 0.2412bh^3 \]
\[ I_{yy} = 0 \quad I_{xx} = \frac{8bh^2}{9\pi} \]
Circular ring (Origin of axes at center.)
Approximate formulas for case when \( t \) is small.

\[
A = 2\pi rt = \pi dt \quad I_x = I_y = \pi r^3 t = \frac{\pi d^2 t}{8}
\]

\( I_{xy} = 0 \quad I_\rho = 2\pi r^3 t = \frac{\pi d^2 t}{4} \)

Semicircle (Origin of axes at centroid.)

\[
A = \frac{\pi r^2}{2} \quad \bar{y} = \frac{4r}{3\pi}
\]

\[
I_x = \frac{(9\pi^2 - 64) r^4}{72\pi} \approx 0.1098 r^4 \quad I_y = \frac{\pi r^4}{8}
\]

\( I_{xy} = 0 \quad I_{BB} = \frac{\pi r^4}{8} \)

Quarter circle (Origin of axes at center of circle.)

\[
A = \frac{\pi r^2}{4} \quad \bar{x} = \bar{y} = \frac{4r}{3\pi}
\]

\[
I_x = I_y = \frac{\pi r^4}{16} \quad I_{xy} = \frac{r^4}{8}
\]

\[
I_{BB} = \frac{(9\pi^2 - 64) r^4}{144\pi} \approx 0.05488 r^4
\]

Quarter-circular spandrel (Origin of axes at vertex.)

\[
A = \left(1 - \frac{\pi}{4}\right) r^2
\]

\[
\bar{x} = \frac{2r}{3(4 - \pi)} \approx 0.7766 r \quad \bar{y} = \frac{(10 - 3\pi)r}{364 - \pi} \approx 0.2234 r
\]

\[
I_x = \left(1 - \frac{9\pi}{16}\right) r^4 \approx 0.01825 r^4 \quad I_y = I_{BB} = \left(1 - \frac{\pi}{16}\right) r^4 \approx 0.1370 r^4
\]

Circular sector (Origin of axes at center of circle.)

\( \alpha = \) angle in radians \( \left( \alpha \leq \frac{\pi}{2} \right) \)

\[
A = \pi r^2 \quad \bar{x} = r \sin \alpha \quad \bar{y} = \frac{2r \sin \alpha}{3\alpha}
\]

\[
I_x = \frac{r^4}{4} (\alpha + \sin \alpha \cos \alpha) \quad I_y = \frac{r^4}{4} (\alpha - \sin \alpha \cos \alpha)
\]

\( I_{xy} = 0 \quad I_\rho = \frac{\alpha r^2}{2} \)